Valuation of the Premium Options for Flexible Premium Equity-Indexed Annuities

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Abstract

In this paper, we extend the single benefit premium evaluation of equity-indexed annuities (EIAs) to periodic annual premiums. Together with various mortality as well as surrender and financial options, these products can be embedded with annual premium options and are then known, in the U.S. market, as flexible premium variable annuities (FPVAs). That is, policyholders are entitled to reinvest every year in the same contract at their convenience. Since the annual investment remains an option, identification of the premium payment schedule that is optimal for the policyholder is a stochastic control problem. Within a discrete model, we solved this stochastic control problem through dynamic programming. Then, we precisely define the distinction between a global and an individual protection of premiums, through some proposed formula for the death and accumulation guarantees. Pursuing the pioneer work of Chi and Lin (2012), we show that the hedging strategy for FPVAs is path dependent, while the hedging strategy for SPVAs is path independent. In the numerical analysis, we detected that the fair participation rate of an FPVA can differ from that of an SPVA by 10% for a global protection of premiums, and by 15% for an individual protection of premiums. The observation of a larger difference for an individual protection, than for a global protection, is a counter-intuitive result and calls for caution when approximating an FPVA by a combination of SPVAs.

Flexible Premium Variable Annuities; Single Premium Variable Annuities; Surrender Value; Rationality; Stochastic Control Problem; Dynamic Programming

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1 Introduction

Equity-indexed annuities (EIAs), now called simply indexed annuities to dispel the presumption that stock market vesting is present in these products, differ from variable annuities (VAs) in these respects: a) the offered investment accounts; and b) in the way that guarantee fees are deducted from the investment account. VAs provide the policyholder with a selection of mutual funds to invest the contributions, while EIAs credit interest to the contributions according to the performance of an equity index, such as the S&P 500. The guarantee fee of an EIA is implicitly expressed by the participation rate in the target index, while the guarantee fee of a VA is explicitly stated as a management fee. Also, for EIAs, the guarantee fee can be implicitly expressed as a limit on the return provided by the index, the so-called \textit{cap}; or a reduction of the return provided by the index, the so-called \textit{spread}.

Apart from these differences, which permit the tailoring of annuity products to a variety of investor profiles, EIAs and VAs are identical in spirit, as they both offer a guaranteed minimum death benefit (GMDB), a guaranteed minimum withdrawal benefit (GMWB), a guaranteed minimum accumulation benefit (GMAB), and the option to completely surrender the contract. See Ledlie et al. (2008) for details on these and other guarantees.

EIAs, like VAs, offer several contribution schemes to the investors. The most frequently encountered scheme in the literature is the single premium variable annuity (SPVA) where the contract is financed through a unique lump-sum payment at inception. In the realm of flexible premium variable annuities (FPVAs), a commonly encountered arrangement is a lump-sum payment at inception, followed by monthly contributions to the contract. FPVAs have been rarely studied in the literature. Apart from the recent work of Chi and Lin (2012), related work has been done on unit-linked insurance contracts with periodic premiums (Schrager & Pelsser, 2004), and participating life insurance contracts (Schmeiser & Wagner, 2011). However, with an FPVA, the policyholder has more freedom for the amount and timing of his periodic premiums, and there is a need for further study of this product. In particular, according to the 2013 IRI Fact book [13], EIAs and VAs sales in 2012 totaled $150 billion dollars. With this macro number, it is not clear what proportion of the sales FPVAs contribute.

Chi and Lin (2012) pioneered the study of FPVAs, within a continuous model for the stock market. They found that the fixed fair guarantee fee of VAs can be up to 45% higher for an FPVA than an SPVA. They also showed how the delta hedging strategy for an FPVA is path dependent, while the delta-hedging strategy for an SPVA is path independent. However, their study was restrained to FPVAs with GMDB and GMAB guarantees where the monthly contribution is pre-determined and fixed; that is, each month, the policyholder re-invests the same amount. The goal of this article is to include the surrender option, and, most importantly, to study the \textit{premium option} embedded in FPVAs: each month, the policyholder has the option to invest or not; if he chooses to re-invest, he has the option to choose the amount of his re-investment. As pointed in Chi and Lin (2012), through the fact sheet of the FPVAs sold by the U.S. company, New York Life, the premium option, as considered here, is in line with currently offered products, more so than the periodic fixed premiums.

The contribution of this article is threefold. First, as far as we know, the premium option of FPVAs has not been analyzed in the literature. We frame the premium option as a stochastic control problem for the policyholder. Then we solve this problem, including the surrender option, within a discrete lattice model, through dynamic programming and the assumption that the policyholder is rational. Second, it is sometimes argued in the literature (Milevsky & Posner, 2001), that FPVAs which offer an individual protection of premiums, as opposed to a global protection of premiums, can be decomposed as a series of SPVAs. The exact meaning of individual versus global protection is yet to be precise. Hence, for the
point-to-point and annual reset class of EIAs, we proposed formulas for the calculation of the GMDB and GMAB guarantees which make clear the distinction between those two types of premium protection. Third, for the point-to-point class of EIAs, we exhibit the hedging strategy for both types of premium protection, and, as in Chi and Lin (2012), we find that the hedging strategy for an FPVA is path dependent, while the hedging strategy for an SPVA is path independent.

A product that is closed, but substantially different from FPVAs, are participating contracts (PCs), as considered in Schmeiser and Wagner (2011), and Bacinello (2003b). Within a PC, the periodic premiums are fixed, and the PC-holder has the right to stop the payment (paid-up option), resume the payments later (resumption option), or surrender the contract completely (surrender option). The combination of the paid-up and resumption option is very close to an FPVA, but each of these options can only be exercised once. In the vocabulary of Steffensen (2002), the options embedded in PCs are intervention options: that is, options that can only be exercised once, but, once exercised, strongly change the payoff. In contrast, the premium option of an FPVA gives the insured a periodic and recurrent right to re-invest or not. Hence, it will be hard to adapt the approach of Schmeiser and Wagner (2011) or Bacinello (2003b). The approach in Steffensen (2002), based on a quasi-variational inequality describing the value of the contract, seems to be adaptable to the premium option of FPVAs. But more work is needed to complete this task, and is left as a subject of future research.

In the case of unit-linked contracts with periodic pre-determined premiums, Schrager and Pelsser (2004) obtained upper and lower bounds for the value of the guarantees by borrowing from Asian options approximation techniques. However, as pointed out by Chi and Lin (2012), the efficiency of those bounds was not proven theoretically. Also, the numerical examples presented in this article show that approximating the participation rate of an FPVA by a combination of SPVAs, whether the premium protection is global or individual, is not reliable. In particular, the fair participation rate of an FPVA can differ from that of an SPVA by 10% for a global protection of premiums, and by 15% for an individual protection of premiums. This result is counter-intuitive, as it is usually argued (Milevsky & Posner, 2001), that approximation by SPVAs works well for individually guaranteed premiums. Hence, additional care must be taken when approximating an FPVA by a combination of SPVAs or when using upper and lower bounds. The direct approach proposed here is one way of asserting the adequacy of these approximations.

This article is structured as follows. Section 2 introduces the discrete stochastic model used. Section 3 describes the stochastic control problem faced by the rational EIA-holder, and proposes a simple algorithm for solving it, which includes both the premium option and the surrender option. Section 4 makes precise the distinction between a global protection of premiums and an individual protection of premiums. Section 5 numerically illustrates the preceding sections and is followed by the conclusion. The code used to perform numerical analysis can be found online at [10].

2 Stochastic Model and Contract Cash Flows

In this section, we present a discrete framework that describes the dynamic of EIA products. Assume that the contract has a maturity of \( n \) years. Let \( \omega = \{\omega_t\}_{t=0}^n \) be the process of the stock for which the moves \( \omega_{t+1} \) take values in some fixed finite set \( W \), with \( \omega_0 = 1 \). Denote by \( \Omega \) the set of all possible stock paths and assume that the level of the stock \( \omega_t \) are recorded at the end of the year. Using a fixed finite set \( W \) for the moves of the stock is equivalent to using a lattice model. Lattice models have been intensively used to model stocks, stock indices, interest rates, and other financial securities due to their flexibility and tractability; see Panjer et al. (1998) and Lin (2006) for example. The model assumes the usual frictionless market: no tax, no transaction cost, etc. The filtration associated with the index process is that generated
by the process. A comprehensive introduction to discrete finance may be found in Pliska (1997).

For the lifetime of the EIA-holder, let \( \tau \) be the first time \( t \) where the payment of the GMDB or GMAB guarantee occurs

\[
\tau = \begin{cases} 
  t & \text{the holder died within } (t-1,t), t = 1, 2, \ldots, n \\
  n & \text{otherwise}
\end{cases}
\]

that is, death can only occur during a year and is reported at the end of the year. Here, the martingale and the physical probability measures are assumed to be the same for mortality. Gaillardetz and Lin (2006) obtained different martingale probability measures for the mortality under certain conditions.

The filtration \( F \) generated by \( \Omega \times T \) corresponds to the direct product of the filtration generated by each process. Assuming that the EIA-holder behaves rationally eliminates the need of stochastically modeling his consumption of the surrender option, and the stochastic dynamic of the contract is completely captured in the probability space \( (\Omega \times T, F, \mathbb{P}) \), where \( \mathbb{P} \) is the direct product of the financial martingale measure and the mortality physical measure. The cash flows of the contract are now defined as \( F \)-adapted processes, that is, at each time \( t \), the order of events is:

1. The index level and the eventual death of the insured is reported at time \( t^- \);
2. The eventual payment from the GMDB or GMAB guarantee is made at time \( t \);
3. In case of survival of the insured, the insured chooses, at time \( t^+ \), between claiming the cash surrender value or investing an additional premium;

where \( t^- \) and \( t^+ \) have the usual meaning.

Define the process \( c = c(\omega, \tau) \) to be one of the possible premium payment schedules allowed by the contract, that is when the state of the world is \( (\omega, \tau) \), the holder’s contributions can be the process \( c \). The contributions are supposed to be paid (or invested) at the beginning of each year,

\[
c_0 = 1, \\
c_t = c_t (\bar{\omega}_t, \tau), t = 1, 2, \ldots, n-1, \\
c_n = 0,
\]

that is, an inception premium is required, and no premium is possible at maturity. Further, assume that each periodic contribution \( c_t, t = 1, 2, \ldots, n-1 \), takes values in some fixed finite set \( C \), and denote by \( \Lambda \) the set of all possible contribution paths. Hence, the contribution scheme allowed by the contract is independent of the market performance, and the space \( \Lambda \times \Omega \) forms a lattice. For example, the single benefit premium of one unit is given by \( c_t = [t = 0] \) and the \( n \)-year level premium of one unit per year is given by \( c_t = [t < \tau] \), with \([\cdot]\) the Iverson bracket. For contracts usually encountered in the market, the set \( \Lambda \) is larger than \( \Omega \times T \); this models the choice offered to the holder. Within this larger set, the holder chooses a strategy \( c \subset \Lambda \) so that, for each path \( (\omega, \tau) \) in \( \Omega \times T \), the holder has a pre-determined \( F \)-adapted contribution schedule \( c(\omega, \tau) \in c \). The rational holder’s stochastic control problem is to identify the strategy \( c \) which maximizes the inception value of the contract.

Let \( b = b(\alpha, \omega, c) \) be the process of the joint level of the GMDB and GMAB accounts with \( \alpha \) the participation rate of the contract. That is, each \( b_t \) represents the guaranteed amount promised to the insured, but is effectively paid only if \( t = \tau \). As explained in Lin and Tan (2003), an EIA is evaluated
through its participation rate \( \alpha \). The benefit payoff needs to be a function of the premiums paid by the holder up to time \( t \)
\[
b_t = b_t (\alpha, \tilde{\omega}, \tilde{c}_{t-1}), \quad t = 1, 2, \ldots, n,
\]
where \( \tilde{\omega} = \{\omega_0, \omega_1, \ldots, \omega_t\} \) and \( b_0 \) is undefined. For \( t < n \), \( b_t \) is the level of the GMDB account, while \( b_n \) is the maturity level of the GMAB account. The possible benefit design for \( b \) will be described with more details in Section 4.

Let \( s = s(\alpha, \omega, c) \) be the process of the surrender value of the contract with
\[
s_0 = s_n = -\infty, \quad s_t = s_t (\alpha, \tilde{\omega}, \tilde{c}_{t-1}) = \theta_t b_t, \quad t = 1, 2, \ldots, n-1,
\]
which means that surrender is not allowed at inception, nor at maturity, and \( 1 - \theta_t \) is the percentage of surrender charge if the policyholder surrenders at time \( t \). As with the benefit account \( b \), the process \( s \) represents the contractual surrender value and not the decision of the holder to surrender the contract. To model the latter, let \( \sigma \) be the \( F \)-stopping time chosen by the holder to surrender his contract. Then, \( \sigma \) takes value in \( \{1, 2, \ldots, n-1\} \), if the surrender option is consumed, and \( \sigma = \infty \) otherwise. Denote by \( \Sigma \) the set of all possible stopping times \( \sigma \). As the next section will show, among all the stopping times in \( \Sigma \), the rational holder chooses the one which maximizes the inception value of his contract.

### 3 Stochastic Control Problem

This section specifies the stochastic control problem faced by the holder of an equity-linked contract with a premium option and a surrender option. In order to describe this control problem, the prospective loss process \( l = l(\alpha, \omega, c) \) underlying the equity-linked products needs to be defined. It represents the difference in the present value at time \( t \) of the insurance benefits minus the periodic premiums given that the policyholder survived to this time. That is the \( c \)-process
\[
l_t = l_t (\alpha, \tilde{\omega}, \tilde{c}_t) = b_t e^{-r(t-t)} [\tau < \sigma] + s_\sigma [\sigma < \tau] e^{-r(\sigma-t)} - \sum_{i=t}^{\min(\tau, \sigma)-1} c_i e^{-r(i-t)}, \quad (3.1)
\]
where \( r \) is the risk free rate and the term \( c \)-process means that \( l \) is \( F \)-adapted but the \( F \)-adapted strategy \( c \) of the policyholder is assumed to be known. Let \( p = p(\alpha, \omega, c) \) denote the \( c \)-process of the fair value (or the reserve). The contract’s fair value is obtained using the expected value of the time \( t \) prospective loss under the martingale measure and is given by
\[
p_t = p_t (\alpha, \tilde{\omega}, \tilde{c}_t) = \mathbb{E}(l_t | F_t, c), \quad t = 0, 1, \ldots, n,
\]
where \( \mathbb{E}(\cdot) \) represents expected value with respect to the measure \( \mathbb{P} \). The stochastic control problem of the rational holder is then to maximize the inception value of the contract
\[
V(\alpha) = \max \{p_0 : c \in \Lambda, \sigma \in \Sigma\}, \quad (3.2)
\]
that is, the rational holder seeks a contribution strategy \( c \) and a stopping time \( \sigma \) which maximizes the inception value \( V(\alpha) \). Dynamic programming is used to secure the optimal solution of (3.2). See Bertsekas (2005) for an introduction to dynamic programming.
Within the discrete setting of this article, dynamic programming turns out to be a less complex task than it may seem at first sight. Indeed, through rationality, the control process in (3.2) can be reduced to \( c \) only, instead of \((c, \sigma)\). As is done in Bacinello (2003a, 2003b), at each time \( t \), the rational holder will compare the cash surrender value with the fair value of the equity-linked contract for the remaining years. If the cash surrender value is equal or greater than the value of the equity-linked contract, he will surrender the policy. This is similar to the valuation of American options where the surrender option is evaluated using a backward recursion. Also, through the independence between financial and mortality risk, the state process can be reduced to \( \Omega \), instead of \( \Omega \times \mathcal{T} \), as solving (3.2) through a backward induction permits one to find the optimal solution given that the insured is alive at each time point.

We now develop the Bellman equation for the stochastic control problem (3.2). Define the set

\[
\mathcal{C}_t = \begin{cases} 
  c_t & t \in \{0, n\} \\
  C & t = 1, 2, \ldots, n-1
\end{cases}
\]

which represents all the admissible premiums at a particular time \( t \). Let \( U_t = U_t(\alpha, \bar{\omega}_t, \bar{c}_{t-1}, C_t) \) denote the time \( t \) fair value given that the holder does not surrender at this time. That is \( U_t \) is a \( C_t \)-functional, which identifies the optimal premium at time \( t \) given the index state \( \bar{\omega}_t \) and the contribution state \( \bar{c}_{t-1} \).

The value \( U_{n-1} \) is the one-year expected discounted payoffs, which, using (3.1), is

\[
U_{n-1} = \max_{c_{n-1} \in \mathcal{C}_{n-1}} \mathbb{E} \left( e^{-r} b_n \mid \bar{\omega}_{n-1}, \bar{c}_{n-1} \right) - c_{n-1},
\]

as the insured seeks to maximize the time \( n-1 \) value of the contract. The equity-linked contract fair value at time \( n-1 \), given that the insured is still alive, is the maximum between the contract value and the cash surrender value

\[
P_{n-1} = \max \{s_{n-1}, U_{n-1}\},
\]

with \( P_t = P_t(\alpha, \bar{\omega}_t, \bar{c}_{t-1}, C_t) \) a \( C_t \)-functional, which, through \( U_t \), produces the optimal contribution \( c_t \) and the component \( (\sigma \mid \bar{\omega}_t, \bar{c}_{t-1}, \tau > t) \) of the optimal stopping time \( \sigma \). To complete the recursive algorithm, suppose that the insured is still alive at time \( t < n-1 \). The time \( t \) fair value \( U_t \), including the surrender option in the future but not at time \( t \), is, using (3.1), given by

\[
U_t = \max_{c_t \in \mathcal{C}_t} \mathbb{E} \left( e^{-r} b_{t+1} \mid \bar{\omega}_t, \bar{c}_t \right) + \mathbb{P}_t \mathbb{E} \left( e^{-r} P_{t+1} \mid \bar{\omega}_t, \bar{c}_t \right) - c_t,
\]

where

\[
a_t = \mathbb{P} (\tau = t + 1 \mid \tau > t) = 1 - \mathbb{P}_t;
\]

since the holder will be entitled to surrender if he survives to time \( t \). Following the same maximization arguments that lead to (3.3), we have

\[
P_t = \max \{s_t, U_t\}, t = 0, 1, \ldots, n-2,
\]

which completes the derivation of the Bellman equation.

The fair inception value \( V(\alpha) \) of the EIA is a function of the participation rate through the payoff process \( b \) and it will be obtained using the equivalence premium principle. This actuarial premium principle considers the insurance contract as a fair game and is based on the possible diversification of the mortality risk. Equations (3.3) and (3.5) combine the actuarial approach and the arbitrage-free theory, as initiated
by Brennan and Schwartz (1976), and Boyle and Schwartz (1977). The fair participation rate $\alpha^*$ is characterized by

$$\alpha^* = \text{root } V(\alpha),$$

(3.6)

with root $\{x \in \mathbb{R} : f(x)\}$ producing a value $x^*$ for which $f(x^*) = 0$. Usually, the root $\{\cdot\}$ operation is performed numerically.

The participation rate $\alpha^*$ is set such that the issuer’s liability, at time zero, is equal to the present value of the periodic optimal premium schedule $c$ that maximizes (3.2). This is a standard application of the equivalence premium principle. However, in practice, a security loading should be added to the fair participation rate in order to protect the issuer against unhedgeable risks, see Gaillardetz and Lakhmiri (2009). Therefore, in the equivalence principle case the insurance company retains $(1 - \alpha^*)$ of the return in order to cover the cost of EIA embedded options. As dynamic programming is used to determine $V(\alpha)$, any possibility of arbitrage is prevented with $\alpha^*$.

As mentioned in the previous section, the discretization of the state space $\Omega$ and the space of the admissible control $\Lambda$ can be done via a lattice, as opposed to being done on a grid. This leads to the tree-based Algorithm 3.7 for calculating $V(\alpha)$. The first step in the algorithm is to construct the space $\Omega \times \Lambda$ as a rooted tree where the branching factor varies in time according to the cardinality of the sets $W \times C_t$. That is, at time $t$, each node $\nu$ of the tree has $|W \times C_t| + 1$ child nodes, with $|W|$ the cardinality of the set $W$. The second step of the algorithm is a backward traversal of the tree which produces $P_0$ as follows. For the terminal node, (3.1) gives the initial condition

$$P_n = b_n.$$ 

Now, fix a node $\nu$ of the tree at time $t < n$. Stratify the child nodes of $\nu$ according to $W$, that is

$$W \times C_{t+1} = \{\eta(\omega_{t+1}) = \omega_{t+1} \times C_t : \omega_{t+1} \in W\}.$$ 

Within the strata $\eta(\omega_{t+1})$, each node has a value for $P_{t+1}$. The node with the highest $P_{t+1}$ corresponds to the optimal fair value of the contract at time $t + 1$, given the filtration of $\nu$ and $\omega_{t+1}$. Using those identified best nodes, together with (3.4) and (3.5), the fair value at node $\nu$ can be obtained through

$$\max \{s_t q_t \mathbb{E}(e^{-r}b_{t+1} | \bar{\omega}_t, \bar{c}_t) + p_t \mathbb{E}(e^{-r}P_{t+1} | \bar{\omega}_t, \bar{c}_t) - c_t\},$$ 

where $\bar{\omega}_t$ and $\bar{c}_t$ are from $\nu$. Indeed, for each $\omega_{t+1}$ in both expectations, $b_{t+1}$ and $P_{t+1}$ are taken from the best node in $\eta(\omega_{t+1})$.

### 4 Financial Guarantees

This section presents some common formulas for the GMDB and GMAB guarantees which where modeled through the process $b$. The presented formulas are centered on EIAs which are priced through a participation rate, but they can be adapted to incorporate a cap or a spread. The purpose of these formulas is to characterize the distinction between FPVAs where the premiums are globally protected versus FPVAs where the premiums are individually protected. The hedging strategy associated with each premium protection type is exhibited and, as in Chi and Lin (2012), shows that the hedging strategy of an FPVA is quite different than that of an SPVA.

The global protection of premiums described is currently offered in the U.S. market (Bauer, Kling & Russ, 2008). But the individual protection of premiums is a proposition of the authors as a natural
Algorithm 3.7 Tree-based solution to the stochastic control problem (3.2) through (3.3) and (3.5)

1: function **Construction of the Tree**
2:     let tree be the result of this construction
3:     set the root node of tree to \((\omega_0, c_0)\)
4:     for \(t \in \{0, 1, \ldots, n - 2\}\):
5:         for each node \(\nu\) of tree at time \(t\):
6:             create the child nodes of \(\nu\) from the set \(W \times C_{t+1}\)
7:     return tree

8: function **Backward Traversal of the Tree to get** \(P_0(tree)\)
9:     for \(t \in \{n - 1, n - 2, \ldots, 0\}\):
10:        for each node \(\nu\) of tree at time \(t\):
11:           for each strata \(\eta(\omega_{t+1})\) in the child nodes of \(\nu\):
12:              identify the node with the highest \(P_{t+1}\)
13:              calculate \(P_t\) for \(\nu\)
14:     return \(P_0\)

extension of the existing global protection of premiums. With these proposed individual guarantees, we numerically analyze if an FPVA can be decomposed as a series of SPVAs, as it is sometimes argued in the literature (Milevsky & Posner, 2001). In the numerical examples, it turns out that this is not the case.

From Lin and Tan (2003), and Tiong (2000), EIA designs may be generally grouped into two broad classes: Annual Reset and Point-to-Point. The index growth on an EIA within the former class is measured and locked in each year. In contrast, the index growth with point-to-point indexing is based on the growth between two time points over the entire term of the annuity. In the case of a periodic premium, the index growth also takes into account the time of the premium.

### 4.1 Point-to-Point

We first consider one of the simplest classes of EIAs, known as the point-to-point. Similarly to variable annuities (equity-linked insurance), the EIA’s payoff needs to be a function of the premiums paid by the policyholder. Hence, the generalized payoff for EIAs could be based on the fund protection guarantees used for variable annuities (see Møller and Steffensen, 2007). The payoff at time \(t + 1\) could then be given by

\[
b^1_{t+1} = \max \left( \sum_{i=0}^{t} c_i \left(1 + \alpha \delta_{i}^{t+1}\right), \beta \sum_{i=0}^{t} c_i (1 + g)^{t+1-i}\right),
\]

where \(g\) is the minimum guaranteed rate of return, \(\beta\) is the participation in the minimum guaranteed return, and the gained rate \(\delta_{i}^{t+1}\) during the interval \([i, t + 1]\) is

\[
\delta_{i}^{t+1} = \frac{\omega_{t+1}}{\omega_{i}} - 1.
\]

The payoff (4.1) provides a global protection of premiums against the loss from a down market \(\beta \sum_{i=0}^{t} c_i (1 + g)^{t+1-i}\). This means that the minimum benefit paid by the issuer is the value of the premiums invested in an account that earns an effective rate of interest of \(g\) at a proportion of 100\(\beta\)%. The nonforfeiture law for Single Premium Deferred Annuities requires that at least 90% of the initial investment be guaranteed and that the rate of return be greater or equal to 3% compounded yearly.
It is important to point out that each premium invested in the equity-indexed annuity contract increases the global level of protection of the premiums without protecting any individual premium, and hence (4.1) offers a global protection of premiums. For an individual protection of premiums based on a point-to-point design, we propose

\[ b_{t+1}^2 = \sum_{i=0}^{t} c_i \max \left( 1 + \alpha \delta_i^{t+1}, \beta (1 + g)^{t+1-i} \right), \quad (4.3) \]

where the financial guarantee is applied to each investment. As confirmed in the numerical examples, the individual protection of premiums (4.3) is more expensive than the global protection (4.1), since the guarantee is applied to each individual investment.

In practice, various designs for \( \delta \) have been proposed. The term-end point design in (4.2) is the simplest crediting method. It measures the index growth from the investment time to the end of a term. The index on the day the premium is invested is taken as the starting index, and the index on the day the policy matures is taken as the ending index.

For this particular design, the payoff \( b_1 \) can be rewritten using an option representation

\[ b_{t+1}^1 = \beta \sum_{i=0}^{t} c_i (1 + g)^{t+1-i} + \alpha \left( \sum_{i=0}^{t} \frac{c_i}{\omega_i} \right) \max \left( \omega_{t+1} - k_1, 0 \right), \]

where

\[ k_1 = \left( \sum_{i=0}^{t} \frac{c_i}{\omega_i} \right)^{-1} \alpha^{-1} \left( \sum_{i=0}^{t} c_i \left( \beta (1 + g)^{t+1-i} + \alpha - 1 \right) \right). \]

The premium protection guarantee \( b_2 \) could also be given by

\[ b_{t+1}^2 = \sum_{i=0}^{t} c_i \left( \beta (1 + g)^{t+1-i} + \alpha \omega_i^{-1} \max \left( \omega_{t+1} - k_2, 0 \right) \right), \]

where

\[ k_i^2 = \omega_i \alpha^{-1} \left( \beta (1 + g)^{t+1-i} + \alpha - 1 \right). \]

These equivalent expressions provide insightful information on hedging strategies that could be used to protect the issuer against both guarantees. They both suggest that hedging strategies should involve a position in zero-coupon bonds and call options with a strike price of either \( k_1 \) or \( k_2 \). Note that these derivatives need to be dealt with at the time of premium payments in order to eliminate the financial risk. It means that at time 0 we need to invest in "forward" bonds and "forward" options that should be sold at time \( i \) with maturity \( t + 1 - i \).

Close observations of \( k_1 \) and \( k_2 \) show that for an FPVA, the hedging strategy must be updated each year according to the performance of the index, while for an SPVA the hedging strategy is independent of the index path. This is in accordance with the result of Chi and Lin (2012), who found, in a continuous time framework, that the hedging strategy for an FPVA is dependent on the path of the index. Also, comparison of \( k_1 \) and \( k_2 \) shows that the hedging strategy for globally protected premiums is different than for individually protected premiums.
4.2 Annual Reset

We now consider the most popular class of EIAs, known as the annual reset. They appeal to investors because they offer similar features as the point-to-point class, with the difference that the interest credited to annual reset EIA contracts cannot be lost. This "lock-in" feature protects the investor against poor performance of the index over a particular year.

The payoff of this type of EIA contract under a global protection of premiums is defined by

\[ b_{t+1}^1 = \max \left( \sum_{i=0}^{t} c_i \prod_{j=i}^{t} \max \left( 1 + \alpha \delta_j, 1 \right), \beta \sum_{i=0}^{t} c_i (1 + g)^{t+1-i} \right), \]  

(4.4)

where \( \delta_j = \delta_j \) represents the realized gained rate in year \( j \), as defined in (4.2).

The payoff of the annual reset class under an individual protection of premiums is naturally given by

\[ b_{t+1}^2 = \sum_{i=0}^{t} c_i \max \left( \prod_{j=i}^{t} \max \left( 1 + \alpha \delta_j, 1 \right), \beta (1 + g)^{t+1-i} \right). \]  

(4.5)

5 Numerical Examples

The code used to produce the example of this section can be found online at [10]. Consider a 5-year EIA issued to an individual aged 55 with a minimum interest rate guarantee of either 3% on 100% of the premiums or 3% on 90% of the premiums. The mortality of the policyholder is assumed to follow the 1979 – 1981 U.S. Life Table (Bowers et al., 1997, Table 3.3.1). The force of interest \( r \) is set to be constant over time and is equal to 6%. The index will be governed by the Cox, Ross, and Rubinstein (1979) model where \( \omega_0 = 1 \) and the number of trading dates \( N \) is 6. In this lattice model, the yearly moves of the index are

\[ W = \{ u^{n-i}d^i \}_{i=0}^{N}, \]

with

\[ u = e^{v/\sqrt{N}} = \frac{1}{d} = 1.107, \]

where \( v = 0.25 \) is the volatility of the index. The martingale probability for the moves of the index is then

\[ \mathbb{P} \left( \frac{\omega_{t+1}}{\omega_t} = u^{n-i}d^i \right) = \binom{n}{i} \pi^{n-i}(1-\pi)^i, \]

for \( t = 0, 1, \ldots, n - 1 \) and \( i = 0, 1, \ldots, N \), with

\[ \pi = \frac{\omega_0 e^r / N - d}{u - d} = 0.524. \]

The analysis is performed using EIAs with term-end point design from the point-to-point and the annual reset classes, for both the global protection \( b_1^1 \) and the individual protection \( b_2^2 \). In order to study the impact of the surrender option, we use the following surrender charges

\[ \theta_t^1 = -\infty, \text{ no surrender option}, \]
\[ \theta_t^2 = 1 - 5%(5 - t), 5\% \text{ per year}, \]
\[ \theta_t^3 = 1 - 1%(5 - t), 1\% \text{ per year}, \]
\[ \theta_t^4 = 100\%, \text{ no surrender penalty}, \]

with \( t = 1, 2, \ldots, 4 \).
5.1 Periodic and Single Premium FPVAs

We begin with an FPVA where the premium payment schedule is set to a single premium, \( c_t = [t = 0] \), or a constant periodic premium, \( c_t = [t < \tau] \), that is, respectively, \( C^{\text{single}} = C = \emptyset \) and \( C^{\text{periodic}} = C = \{1\} \). Table 1 gives the participation rates obtained using (3.6) and Algorithm 3.7.

<table>
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<th>( \beta %)</th>
<th>( \theta^1 )</th>
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<th>( \theta^2 )</th>
<th>Periodic</th>
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Table 1: The participation rates under constant periodic premiums and single premium

From table 1, an individual protection of premiums provides lower participation rates, meaning that it is more expensive than a global protection of premiums.

The impact of the premium payment schedule on the participation rate can be explained by two different phenomena. Firstly, a periodic premium contract should be more costly than a corresponding single premium contract, as, within the former, the steady stream of premiums generates a higher benefit payoff. On the other hand, the periodic investment could decrease the variability of the payoff functions because of the premiums at the end of the contract. This should reduce the cost of equity-indexed products. Both behaviors could be observed in our numerical examples.

It is also important to emphasize that a greater impact on the participation rates is observed for the point-to-point than for the annual reset since it already protects each annual return from the index. For the surrender options, a contract with a smaller surrender charge \( \beta \) is more expensive than a similar contract with a higher surrender charge. This is to cover the cost of the surrender option. The surrender option reduces the difference between the participation rates under periodic and single premiums and also between the global protection and the individual protection of premiums.

5.2 The Premium Option of FPVAs

We now investigate a premium option of FPVAs which allows a policyholder, each period, to choose between two distinct premium payments; that is, FPVAs with \( C^1 = C = \{0, 1\} \), or \( C^2 = C = \{0.5, 1.5\} \) both with \( c_0 = 1 \). These types of FPVAs give rise to a stochastic control problem, and their consideration is an extension of the work of Chi and Lin (2012). Although the consideration of only two choices for the periodic premium payment may seem restrictive, through the assumption of rationality of the policyholder and the benefit payoff functions used in this article, this is not the case. Indeed, as all the financial guarantees of Section 4 are monotone functions of the premium schedule, the benefit payoff \( b_{t+1} \) has exactly one absolute minimum, and exactly one absolute maximum, given \( \bar{\omega}_t \) and \( \bar{c}_{t-1} \). Hence, the consideration
of the set $C^1$ is equivalent to solving the stochastic control problem as if any yearly premium within the interval $[0, 1]$ is admissible.

For $C^1$, in Table 2, EIAs with global protection of premiums $b^1$ are more expensive than the respective rates for the individual protection of premiums $b^2$. The freedom allowed by the premium option also reduces the impact on participation rates of the surrender options. The effect on the participation rate is limited for the point-to-point class and negligible for the annual reset. This is because of the "ratchet" type of financial guarantees embedded in the annual reset class.

For $C^2$, in Table 3, the situation is similar. The $C^1$-contract could be more expensive, e.g. the annual reset with $b^2$ and $\beta = 100$, or cheaper, e.g. point-to-point with $b^1$ and $\beta = 100\%$, than the $C^2$-contract. Moreover, the $C^2$-point-to-point contract with $b^1$ and $\beta = 100\%$ could be less or more expensive than the $C^1$-contract depending on the surrender charges.

The $C^1$ contract deviates from what is usually offered in the U.S., as a minimum monthly contribution is usually required. But, a comparison of Tables 2 and 3 shows that the costs of those contracts are very similar.

The $C^1$ and $C^2$ contracts are always more expensive than the corresponding $C^{\text{periodic}}$, which is itself more expensive than the $C^{\text{single}}$ contract. For a global protection of premiums, the highest deviation of 10% from $C^{\text{single}}$ is observed with both the $C^1$ and $C^2$ point-to-point contracts. While, for an individual protection of premiums, the highest deviation of 15% from $C^{\text{single}}$ is also observed with both the $C^1$ and $C^2$ point-to-point contracts. In contrast, the annual reset contracts deviate from at most 5% of the corresponding $C^{\text{single}}$ contract.

The observed deviation for point-to-point contracts is counter-intuitive (Milevsky & Posner, 2001), as the global protection of premiums triggers a smaller deviation to $C^{\text{single}}$ than the individual protection of premiums. Hence, in the presence of a premium option, approximating an FPVA by a combination of SPVs, or valuating an FPVA through the usage of upper and lower bounds, requires additional precautions.

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Table 2: The participation rates for $C = \{0, 1\}$

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Table 3: The participation rates for $C = \{0.5, 1.5\}$
6 Conclusions

The purpose of this paper was to evaluate the premium option that can be embedded in equity-linked products, together with the surrender option. To this end, a discrete stochastic model is introduced, together with the stochastic control problem faced by the policyholder. Through rationality and independence between financial and biometric risk, we developed a simple expression for the Bellman equation, together with a tree-based algorithmic implementation of the Bellman equation.

The distinction between a global protection of premiums and an individual protection of premiums was made precise through proposed formulas for the GMDB and GMAB guarantees. The hedging strategy for the point-to-point class of EIAs was shown to be path dependent for both a global and an individual protection of premiums. Meanwhile, the hedging strategy for an SPVA is path independent.

Numerically, we found that the fair participation rate of an FPVA can differ from that of an SPVA by 10% for a global protection of premiums, and by 15% for an individual protection of premiums. This result is counter-intuitive, and warns for additional caution when approximating an FPVA by a combination of SPVAs.

The analysis was performed using a discrete lattice model and dynamic programming, a context which particularly attracts the curse of dimensionality, especially for long term contracts. A possible avenue of future research is the conception of valuation algorithms for FPVAs, which more easily accommodate long term contracts.

Acknowledgments

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References


